UNIT-V

multi variable calculus (Partial Differentiation & applications):-

Higher order Partial Differential Equation:-

Definition of PDE:- A D.E having more than one independent variable is called a "PDE."

Eg:- (1). $f(x,y) = x^2+y^2$

Here $f_x = \frac{\partial f}{\partial x} = 2x.$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} = 2. = \frac{\partial^2 f}{\partial x^2}$

$f_y = \frac{\partial f}{\partial y} = 2y$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy} = 2. = \frac{\partial^2 f}{\partial y^2}$

$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(2y) = 0.$

(2). $f(x,y) = \log(x^2y^2)$

$= \log x^2 + \log y^2$

$f_x = \frac{2x}{x^2} = \frac{2}{x}$

$f_y = \frac{2}{y}$

$f_{xx} = \frac{-2}{x^2}$

$f_{yy} = \frac{-2}{y^2}$

$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{2}{y}\right) = 0$

Limit of a f^n with two variables:-

The variable value $f(x,y)$ approaches a finite fixed value 'l' in the co-domain when the variable value (x,y) approaches a fixed value (a,b) i.e, $x \to a$, $y \to b$ simultaneously then we write it as $\lim\limits_{\substack{x\to a\\ y\to b}} f(x,y) = \lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b) = l.$

$\lim\limits_{(x,y)\to(a,b)} f(x,y) = l$ (or) $\Rightarrow \lim\limits_{x\to a} f(x,b) = \lim\limits_{y\to b} f(a,y) = l.$

(or)

$$\lim_{x\to a}\left\{\lim_{y\to b} f(x,y)\right\} = \lim_{y\to b}\left\{\lim_{x\to a} f(x,y)\right\} = l$$

— * Continuity of a f^n of two variables $f(x,y)$ is said to be continuous at (a,b) of it's domain of definition

if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

i.e If a function is continuous at each point of it's domain then we say that it is continuous in that domain.

— * A function which is not continuous is called discontinuous.

(1). Evaluate $\lim_{\substack{x\to 1\\ y\to 2}} \dfrac{2x^2y}{x^2+y^2+1}$

Sol^n:- $\lim_{x\to 1}\left\{\lim_{y\to 2} \dfrac{2x^2y}{x^2+y^2+1}\right\} = \lim_{x\to 1}\left\{\dfrac{4x^2}{x^2+5}\right\} = 2/3$.

(or)

$\lim_{y\to 2}\left\{\lim_{x\to 1} \dfrac{2x^2y}{x^2+y^2+1}\right\} = \lim_{y\to 2}\left\{\dfrac{2y}{y^2+2}\right\} = 2/3$.

(2). $f(x,y) = \dfrac{x-y}{2x+y}$ then P.T $\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\} \neq \lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\}$

Sol^n L.H.S $= \lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\} = \lim_{x\to 0}\left\{\lim_{y\to 0} \dfrac{x-y}{2x+y}\right\} = \lim_{x\to 0}\left(\dfrac{x}{2x}\right)$

$= 1/2$

R.H.S $= \lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\} = \lim_{y\to 0}\left(\dfrac{-y}{y}\right) = -1$.

$\therefore$ LHS $\neq$ RHS.

3. Discuss the continuity of the function

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$$

Sl^n:- $\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\} = \lim_{x\to 0}\left\{\lim_{y\to 0}\dfrac{2xy}{x^2+y^2}\right\} = 0.$

$\lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\} = \lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\} = 0.$

$\therefore$ $f(x,y)$ is continuous at $x=0, y=0$

We will check along $y = mx$

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0}\frac{2xy}{x^2+y^2} = \lim_{x\to 0}\frac{2x(mx)}{x^2+m^2x^2} = 0$$

$$\lim_{y\to 0} f(x,y) = \lim_{y\to 0}\frac{2xy}{x^2+y^2} = \lim_{y\to 0}\frac{2x(mx)}{x^2+m^2x^2} = \frac{2x^2m}{x^2+x^2m^2}$$

(or)

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\substack{x\to 0\\ y\to 0}}\frac{2xy}{x^2+y^2}$$

(or) $\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\} = \lim_{x\to 0}\left\{\lim_{y\to mx}\dfrac{2xmx}{x^2+m^2x^2}\right\}$

$= \dfrac{2m}{1+m^2}$

$$= \frac{2x^2m}{x^2+x^2m^2} = \frac{2m}{1+m^2}$$

$\lim_{y\to mx}\left\{\lim_{x\to 0} f(x,y)\right\} = \lim_{y\to 0}\lim_{x\to 0}\left\{\dfrac{2xmx}{x^2+m^2x^2}\right\}$

$= 2m/1+m^2$

which is different for the different values of 'm' selected

$\therefore$ $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does not exist.

4. Examine for the continuity at the origin of the function defined by $f(x,y) = \dfrac{x^2}{\sqrt{x^2+y^2}}$ for $x\neq 0, y\neq 0$;

for $x=0$, $y=0$

Redefine the f^n to make it continuous.

$$\lim_{x\to0}\left\{\lim_{y\to0} f(x,y)\right\} = \lim_{x\to0}\lim_{y\to0}\left\{\frac{x^2}{\sqrt{x^2+y^2}}\right\} = 0.$$

$$\lim_{y\to0}\left\{\lim_{x\to0} f(x,y)\right\} = \lim_{y\to0}\left\{\lim_{x\to0}\frac{x^2}{\sqrt{x^2+y^2}}\right\} = 0.$$

$\therefore$ $f(x,y)$ is continuous at $x=0, y=0$

We will check along $\underline{y=mx}$

$$\lim_{y\to0}\left\{\lim_{x\to0} f(x,y)\right\} = \lim_{y\to0}\left\{\lim_{x\to0}\frac{x^2}{\sqrt{x^2+y^2}}\right\} = 0. \quad \text{(or)} \quad \lim_{x\to0}\left\{\frac{x^2}{\sqrt{x^2+m^2x^2}}\right\} = 0$$

$$\lim_{y\to mx}\{0\} = 0.$$

$$\lim_{x\to0}\left\{\lim_{y\to0} f(x,y)\right\} = \lim_{x\to0}\left\{\lim_{y\to0}\frac{x^2}{\sqrt{x^2+y^2}}\right\} = \lim_{x\to0}\frac{x^2}{\sqrt{x^2+m^2x^2}} = 0.$$

$\therefore$ $f(x,y)$ is continuous at origin O.

5. $\frac{\partial^2}{\partial x\partial y}\left(e^{xy}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} e^{xy}\right)$$

$$= \frac{\partial}{\partial x}\left\{e^{xy}\left[x+0\right]\right\}$$

$$= \frac{\partial}{\partial x}\left(e^{xy} x\right)$$

$$= e^{xy}(1) + x e^{xy}(y)$$

$$= e^{xy}(1+yx)$$

6. $\frac{\partial}{\partial y}\left(a^{x^2y^2}\right)$

$$= a^{x^2y^2}\log a \, \frac{\partial}{\partial y}(x^2y^2)$$

$$= a^{x^2y^2}\log a \,(2yx^2)$$

7. $\frac{\partial}{\partial x} e^{x^y} = \frac{\partial}{\partial x} e^{x^y} = e^{x^y}\cdot\frac{\partial}{\partial x}(x^y) = e^{x^y}\left(yx^{y-1}\right)$

$$= yx^{y-1}e^{x^y}$$

8. $\frac{\partial}{\partial y}\left(e^{x^y}\right)$

$$\Rightarrow e^{x^y}\frac{\partial}{\partial y}(x^y)$$

$$\Rightarrow e^{x^y}x^y\log x$$

(9). find $\frac{\partial^2}{\partial x^2}(\sin xy)$

$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(\sin xy)\right]$

$= \frac{\partial}{\partial x}\left[(\cos xy)y\right]$

$= y(-\sin xy)y$

$= -y^2 \sin xy$

(10). $\frac{\partial^2}{\partial x \partial y}(\sin xy)$

$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\sin xy\right)$

$= \frac{\partial}{\partial x}\left(\cos xy\,(x)\right)$

$= \cos xy + x(-\sin xy)y$

$= -xy \sin xy + \cos xy.$

(11). Find the first and second order partial derivatives for the f^n $f(x,y) = \log(x^2+y^2)$

Sol^n.- $\Rightarrow \frac{\partial f}{\partial x} = f_x = \frac{1}{x^2+y^2}(2x) = \frac{2x}{x^2+y^2}$

$\Rightarrow f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right) = \frac{2(x^2+y^2) - 2x(2x)}{(x^2+y^2)^2} = \frac{2x^2+2y^2-4x^2}{(x^2+y^2)^2}$

$= \frac{2y^2-2x^2}{(x^2+y^2)^2}.$

$\Rightarrow \frac{\partial f}{\partial y} = f_y = \frac{\partial}{\partial y}(\log(x^2+y^2)) = \frac{2y}{x^2+y^2}$

$\Rightarrow f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right) = \frac{2(x^2+y^2) - (2y)(2y)}{(x^2+y^2)^2}. = \frac{2x^2-2y^2}{(x^2+y^2)^2}$

(12). P.T $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ of $u = \sin x \sin y$.

L.H.S $= \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}(\sin x \sin y)\right] = \frac{\partial}{\partial y}\left[\sin y \cos x\right] = \cos x \cos y$

R.H.S $= \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left[\sin x \cos y\right] = \cos y \cos x$

$\therefore$ L.H.S = R.H.S

(13). If $x = r\cos\theta$; $y = r\sin\theta$ then P.T $r_{xx} + r_{yy} = \frac{1}{r}$

Solⁿ:- $x^2 = r^2\cos^2\theta$; $y^2 = r^2\sin^2\theta$

$\therefore x^2 + y^2 = r^2$

$\Rightarrow r = \sqrt{x^2+y^2}$.

$$r_x = \frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$r_{xx} = \frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \sqrt{x^2+y^2} - x\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} \Big/ \left(\sqrt{x^2+y^2}\right)^2$$

$$= \frac{\dfrac{x^2+y^2-x^2}{\sqrt{x^2+y^2}}}{(x^2+y^2)} = \frac{y^2}{(x^2+y^2)^{3/2}} \quad \text{---(1)}$$

$$r_y = \frac{\partial r}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

$$r_{yy} = \frac{\partial^2 r}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial r}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = \sqrt{x^2+y^2} - \frac{y}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}y) \Big/ x^2+y^2$$

$$= \frac{x^2+y^2-y^2}{\sqrt{x^2+y^2}} \Big/ x^2+y^2 = x^2 \Big/ (x^2+y^2)^{3/2} \quad \text{---(2)}$$

$\therefore r_{xx} + r_{yy} = (1) + (2)$

$$= \frac{y^2}{(x^2+y^2)^{3/2}} + \frac{x^2}{(x^2+y^2)^{3/2}}$$

$$= \frac{(x^2+y^2)}{(x^2+y^2)^{3/2}} \qquad = \frac{\cancel{x^2+y^2}}{\cancel{(x^2+y^2)}(x^2+y^2)^{1/2}}$$

$$= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r} \;//$$

(14) (i). If $f(x,y) = x\cos y + y\cos x$ then
(ii). $x^3+y^3-3axy = f(x,y)$.
then P.T $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

(i):
Sol:- $L.H.S = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$= \frac{\partial}{\partial x}\frac{\partial}{\partial y}(x\cos y + y\cos x)$ [∵ (i)]

$= \frac{\partial}{\partial x}(-x\sin y + \cos x)$

$= -\sin y - \sin x = -(\sin x + \sin y)$

$R.H.S = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(\cos y - y\sin x)$

$= -\sin y - \sin x = (\sin y + \sin x)$

∴ L.H.S = R.H.S.

(ii). $L.H.S = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(3y^2 - 3ax) = -3a$

$R.H.S = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(3x^2 - 3ay) = -3a$.

∴ L.H.S = R.H.S.

(15). If $f(x,y) = ax^3 + hx^2y + by^3$ then find $f_x, f_{xx}, f_y, f_{yy}, f_{xy}, f_{yx}$.

(16). Verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for the function $u = \tan^{-1}(x/y)$

Chain rule of Partial Derivatives :-

Let $f = f(u, v)$ and $u = u(x, y)$
$v = v(x, y)$

then $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}$$

Note :- Let $f = f(u, v, w)$ and $u = u(x, y, z)$
$v = v(x, y, z)$
$w = w(x, y, z)$

then $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial y}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial z}$$

Note : Let $f = f(u, v, w)$ and $u = u(x, y)$
$v = v(x, y)$
$w = w(x, y)$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x}.$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial y}.$$

Note :- Let $f = f(u, v)$ and $u = u(x, y, z)$
$v = v(x, y, z)$.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}$$

& $$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial z}.$$

Total derivatives :-

Let $f(u,v)$ & 'u' is a f^n of (x) & $u(x)$

'v' is a f^n of (x). $v(x)$.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}.$$

①. If $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$ then P.T

$$\frac{1}{2}\cdot\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0.$$

Sol^n:- $u = f(r,s,t)$ where $r = 2x-3y$; $s = 3y-4z$, $t = 4z-2x$

$r = r(x,y)$ $\quad s = s(y,z)$ $\quad t = t(z,x)$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= f_r(2) + f_s(0) + f_t(-2)$$

$$= 2f_r - 2f_t$$

$$= 2(f_r - f_t) \quad \text{—— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= f_r(-3) + f_s(3) + f_t(0)$$

$$= -3f_r + 3f_s \quad \text{—— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= f_r(0) + f_s(-4) + f_t(4)$$

$$= -4f_s + 4f_t \quad \text{—— (3)}$$

$$\therefore L.H.S = \frac{1}{\cancel{2}}\cancel{2}(f_r - f_t) + \frac{1}{\cancel{3}}\cancel{3}(f_s - f_r) + \frac{1}{\cancel{4}}\cancel{4}(f_t - f_s)$$

$$= \cancel{f_r} - \cancel{f_r} - \cancel{f_t} + \cancel{f_t} + \cancel{f_s} - \cancel{f_s} = 0$$

$$= 0$$

R.H.S.

②. If $u = u(y-z,\ z-x,\ x-y)$ then S.T $u_x + u_y + u_z = 0$

Sol: let $u = u(r, s, t)$. $r = y-z,\ s = z-x,\ t = x-y$.

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= u_r(0) + u_s(-1) + u_t(1) \Rightarrow u_x = -u_s + u_t$$

$$u_y = u_r - u_t$$

$$u_z = -u_r + u_s$$

$\therefore$ $LHS = u_x + u_y + u_z = -\cancel{u_s} + \cancel{u_t} + \cancel{u_r} - \cancel{u_t} - \cancel{u_r} + \cancel{u_s}$

$= 0. = R.H.S$

③. If $u = f(r, s, t)$ where $r = x/y,\ s = y/z,\ z/x = t$ then P.T

$x u_x + y u_y + z u_z = 0.$

Sol:- $\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x}$

$u_x = f_r\left(\frac{1}{y}\right) + f_s(0) + f_t\left(\frac{-z}{x^2}\right)$

$\therefore x u_x = \frac{x}{y} f_r - \frac{xz}{x^2} f_t$

$x u_x = \frac{x}{y} f_r - \frac{z}{x} f_t$ —— (1)

$y u_y = -\frac{x}{y} f_r + f_s \frac{y}{z}$ —— (2)

$z u_z = -f_s \frac{y}{z} + f_t \frac{z}{x}$ —— (3)

(1) + (2) + (3).

$x u_x + y u_y + z u_z = 0.$

4. If $u = f(x^2-y^2, y^2-z^2, z^2-x^2)$ then P.T $\frac{1}{x}u_x + \frac{1}{y}u_y + \frac{1}{z}u_z = 0$ (11)

Sol:- let $u = f(r, s, t)$ $\quad r = x^2+y^2$; $s = y^2-z^2$; $t = z^2-x^2$

$$\frac{1}{x}u_x = 2f_r - 2f_t \quad \text{——— (1)}$$

$$\frac{1}{y}u_y = -2f_r + 2f_s \quad \text{——— (2)}$$

$$\frac{1}{z}u_z = -2f_s + 2f_t \quad \text{——— (3)}$$

(1) + (2) + (3)

$$\therefore \frac{1}{x}u_x + \frac{1}{y}u_y + \frac{1}{z}u_z = 0.$$

L.H.S = R.H.S.

5. If $u = x/z$, $v = y/z$, $w = z$ and $f = f(u, v, w)$ then P.T
$x f_x + y f_y + z f_z = w \cdot f_w$

Sol:- $f_x = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}$

$$= f_u\left(\frac{1}{z}\right) + f_v(0) + f_w(0)$$

$$x f_x = \frac{x}{z} f_u \quad \text{——— (1)}$$

$$y f_y = \frac{y}{z} f_v \quad \text{——— (2)}$$

$$z f_z = -\frac{x}{z} f_u - \frac{y}{z} f_v + z f_w \quad \text{——— (3)}$$

(1) + (2) + (3)

$$x f_x + y f_y + z f_z = z f_w.$$

$$= w f_w \quad // \quad (\because z = w).$$

6. If $x = r\cos\theta$ & $y = r\sin\theta$ then P.T

(i). $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$

(ii). $\frac{1}{r}\frac{\partial x}{\partial \theta} = r\frac{\partial \theta}{\partial x}$

Sol:- (i). $x = r\cos\theta$; $y = r\sin\theta$

$x^2 = r^2\cos^2\theta$; $y^2 = r^2\sin^2\theta$

$\therefore x^2+y^2 = r^2$ & $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \Rightarrow \theta = \tan^{-1}(y/x)$

$r = \sqrt{x^2+y^2}$

L.H.S.

(i). $\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}(2x) = \frac{x}{\sqrt{x^2+y^2}} = \frac{\cancel{r}\cos\theta}{\cancel{r}} = \cos\theta.$

R.H.S $= \frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r\cos\theta) = \cos\theta.$

$\therefore$ L.H.S = R.H.S.

(ii). $\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(r\cos\theta) = -r\sin\theta$

L.H.S $= \frac{1}{r}\frac{\partial x}{\partial \theta} = -\sin\theta$

$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}(\tan^{-1}y/x) = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} = \frac{-\cancel{r}\sin\theta}{r^{\cancel{2}}} = \frac{-\sin\theta}{r}$

R.H.S $= r\frac{\partial \theta}{\partial x} = \cancel{r}\left(\frac{-\sin\theta}{\cancel{r}}\right) = -\sin\theta$

$\therefore$ L.H.S = R.H.S.

Euler's Theorem :

Homogenous function :- A function $f(x,y)$ is said to be a Homogenous of degree (or) order 'n' in variables x, y if $f(kx, ky) = k^n f(x,y)$, where 'n' is a real number.

Eg: (1). $f(x,y) = \dfrac{x^2+y^2}{x^2-y^2}$

$$f(kx, ky) = k^0\left(\frac{x^2+y^2}{x^2-y^2}\right)$$

$$= k^n f(x,y).$$

where $n=0$.

$\therefore f(x,y)$ is a homogenous function with degree (or) order zero.

Eg (2):- $f(x,y) = \dfrac{x^2+y^2}{x^3-y^3}$.

$$f(kx, ky) = \frac{k^2x^2+k^2y^2}{k^3x^3-k^3y^3}$$

$$= k^{-1}\left(\frac{x^2+y^2}{x^3-y^3}\right)$$

$$= k^{-1} f(x,y)$$

where $n=-1$

$\therefore f(x,y)$ is a homogenous f^n with degree (or) order -1

3. $f(x,y) = \frac{x^4+y^4}{x^2+y^2}$

$f(kx,ky) = \frac{k^4x^4+k^4y^4}{k^2x^2+k^2y^2} = k^2\left(\frac{x^4+y^4}{x^2+y^2}\right)$

Here $n=2$ $\therefore$ order $=2$.

4. $f(x,y) = x^{1/3}y^{3/4}\tan^{-1}(y/x)$

$f(kx,ky) = k^{1/3}x^{1/3}\cdot y^{3/4}k^{3/4}\tan^{-1}\left(\frac{ky}{kx}\right)$

$= k^{13/12}x^{1/3}y^{3/4}\tan^{-1}(y/x)$

Euler's theorem on Homogenous function :-

Statement :- If $z=f(x,y)$ is a Homogenous function of degree 'n', then $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = nz$, $\forall$ x,y in the domain of the f^n.

Proof :- $z=f(x,y)$ is a homogenous function of degree 'n' in 'x' & 'y' then

$z = x^n g(y/x)$ —— (1).

(1) w.r.to 'x'.

$\frac{\partial z}{\partial x} = nx^{n-1}g(y/x) + x^n g'(y/x)\left(\frac{-y}{x^2}\right)$

$x\frac{\partial z}{\partial x} = \left[-y\,x^{n-2}g'(y/x) + nx^{n-1}g(y/x)\right]x$

$\Rightarrow x\frac{\partial z}{\partial x} = -yx^{n-1}g'(y/x) + nx^n g(y/x)$ —— (2).

(1) w.r.to 'y'

$z = x^n g(y/x)$

$\frac{\partial z}{\partial y} = x^n g'(y/x)\left(\frac{1}{x}\right) + 0 \Rightarrow y\frac{\partial z}{\partial y} = yx^{n-1}g'(y/x)$ —— (3)

(2) + (3).

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \cancel{-yx^{n-1}g'(y/x)} + nx^n g(y/x) + \cancel{yx^{n-1}g'(y/x)}$$

$$= nx^n g(y/x)$$

$$= n \cdot z \quad (\because (1))$$

$$\therefore \boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz}$$

Hence Proved.

Note:- If 'u' is a homogenous function of 'x','y' & 'z' of degree 'n'

then $\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu.}$

(1). If $f(x,y) = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ then find $x.\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$.

Sl^n:- $f(kx, ky) = \frac{\sqrt{kx}-\sqrt{ky}}{\sqrt{kx}+\sqrt{ky}} = \frac{k^{1/2}(\sqrt{x}-\sqrt{y})}{k^{1/2}(\sqrt{x}+\sqrt{y})} = k^0 f(x,y)$.

$\therefore f(x,y)$ is a homogenous f^n with degree zero $= n$.

$$\therefore x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n.f$$

$$\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0.$$

(2). If $u = \left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 + \left(\frac{x}{y}\right)$ then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

Sl^n:- $u(kx, ky, kz) = \left(\frac{kx}{kz}\right)^2 + \left(\frac{ky}{kz}\right)^2 + \left(\frac{kx}{ky}\right)$

$$= k^0\left[\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 + \left(\frac{x}{y}\right)\right]$$

$$= k^0 u(x, y, z)$$

Here $n = 0$.

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = 0$$

③. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$, if $u=\frac{x^3y^3}{y^3+x^3}$.

Sol:- $u(kx,ky)=\frac{k^3x^3 k^3y^3}{k^3y^3+k^3x^3}=\frac{k^6x^3y^3}{k^3(y^3+x^3)}=k^3[u(x,y)]$

Here $n=3$.

$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$
$=3u$

④. If $u=\frac{x^2y^2}{x+y}$.

Sol:- $u(x,y)=\frac{x^2y^2}{x+y}$

$u(kx,ky)=\frac{k^2x^2k^2y^2}{kx+ky}=\frac{k^4x^2y^2}{k(x+y)}=k^3u(x,y)$ Here $n=3$.

$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu=3u$

—* If $u(x,y)=\log\left(\frac{x^4+y^4}{x+y}\right)$, S.T $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$.

Sol:- $e^u=\frac{x^4+y^4}{x+y}=f(x,y)$

$f(kx,ky)=k^3f(x,y)$, $n=3$.

$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$ ~~=~~ $x\frac{\partial}{\partial x}e^u+y\frac{\partial}{\partial y}e^u=3e^u$

$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$

—* If $u=\tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ S.T $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$

Sol:- $\tan u=\frac{x^3+y^3}{x+y}=f(x,y)$

$\therefore x\frac{\partial}{\partial x}\tan u+y\frac{\partial}{\partial y}\tan u=3\tan u$

$\Rightarrow x\sec^2u\frac{\partial u}{\partial x}+y\sec^2u\frac{\partial u}{\partial y}=3\tan u$

$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$

③ Verify Euler's theorem for $z=ax^2+2hxy+by^2$.

Sol:- Here $z=z(x,y)$

by Euler's theorem $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$ —— (1).

Verification:-

L.H.S $=x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x(2ax+2hy)+y(2by+2hx)$

$=2ax^2+2hxy+2by^2+2hxy$

$=2(ax^2+by^2+2hxy)$

$=2z$.

R.H.S $=nz$

$=2z$

Hence Proved.

[$\because z(kx,ky)=ak^2x^2+2hkx.ky+bk^2y^2$

$=k^2(ax^2+2hxy+by^2)$

$=k^2z(x,y)$ where $n=2$]

(6). $u = f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$ then P.T $\Sigma x \cdot \frac{\partial u}{\partial x} = -u$

Sol:- $u(kx, ky, kz) = (k^2x^2+k^2y^2+k^2z^2)^{-1/2}$

$= (k^2)^{-1/2}(x^2+y^2+z^2)^{-1/2}$

$= k^{-1}\, u(x,y,z)$

where $n = -1$.

by Euler's thm $\Sigma x \frac{\partial u}{\partial x} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$

$= -u = $ R.H.S.

L.H.S $= \Sigma x \frac{\partial u}{\partial x} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

$= x\left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2x) + y\left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2y) + z\left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2z)$

$= -x^2(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-3/2}(-y^2) + (x^2+y^2+z^2)^{-3/2}(-z^2)$

$= (-1)(x^2+y^2+z^2)^{-3/2}(x^2+y^2+z^2)^{1}$

$= -(x^2+y^2+z^2)^{-1/2}$

$= -u.$

$\therefore$ R.H.S = L.H.S

(7). Verify Euler's thm $u = \sin^{-1}(x/y) + \tan^{-1}(y/x)$

Sol:- $u = u(x,y) = \sin^{-1}(x/y) + \tan^{-1}(y/x)$.

by Euler's thm $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$ — (1)

L.H.S $= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left[\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{1}{y}\right) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)\right] + y\left[\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{-x}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)\right]$

$= x\left[\frac{\cancel{y}}{\sqrt{y^2-x^2}}\frac{1}{\cancel{y}} + \frac{\cancel{x^2}}{x^2+y^2}\left(-\frac{y}{\cancel{x^2}}\right)\right] + y\left[\frac{y}{\sqrt{y^2-x^2}}\left(\frac{-x}{y^2}\right) + \frac{x^2}{x^2+y^2}\frac{1}{x}\right]$

$$= \cancel{\frac{x}{\sqrt{y^2-x^2}}} - \cancel{\frac{xy}{x^2+y^2}} \quad \cancel{-\frac{x}{\sqrt{y^2-x^2}}} + \cancel{\frac{xy}{x^2+y^2}} = 0.$$

R.H.S $= nu$

$= 0.$

$$\left[\because u(kx, ky) = \sin^{-1}\left(\frac{kx}{ky}\right) + \tan^{-1}\left(\frac{ky}{kx}\right)\right.$$

$$= k^0\left[\sin^{-1}(x/y) + \tan^{-1}(y/x)\right] = k^0 u(x,y)$$

where $n = 0$

$\therefore$ L.H.S = R.H.S

Jacobian :- Let $u = u(x,y)$ & $v = v(x,y)$ then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \text{ (or) } \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

is called the Jacobian of u, v w.r.to x, y (or) the Jacobian of the transformation.

It is denoted by $J\left[\frac{u,v}{x,y}\right]$ (or) $\frac{\partial(u,v)}{\partial(x,y)} = J\frac{(u,v)}{(x,y)}$

$\boxed{\text{Note}}$:- If $u = u(x,y,z)$ and $v = v(x,y,z)$ and $w = w(x,y,z)$

then

$$J\frac{(u,v,w)}{(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Properties :- ① $J\left(\frac{u,v}{x,y}\right) \cdot J\left[\frac{x,y}{u,v}\right] = 1.$

— * If $u(x,y) = \log\left(\frac{x^4+y^4}{x+y}\right)$, S.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$.

Sol:- Given that $u = \log\left(\frac{x^4+y^4}{x+y}\right)$ — (1)

Since 'u' is not a homogenous function, we write (1) as

$$e^u = \frac{x^4+y^4}{x+y} = f(x,y)$$

$$\Rightarrow u(kx, ky) = \frac{(kx)^4 + (ky)^4}{kx+ky} = k^3\left(\frac{x^4+y^4}{x+y}\right)$$

$$\Rightarrow u(kx, ky) = k^n u(x,y)$$

where $n=3$.

$\therefore$ by Euler's thm $\quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

$$\Rightarrow x\frac{\partial}{\partial x}e^u + y\frac{\partial}{\partial y}e^u = 3e^u$$

$$\Rightarrow xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = 3e^u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3.$$

//

$\boxed{\text{Note}}$: Euler's theorem of second order differential Eq^n is

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\left[g'(u) - 1\right]$$

— * Find $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$, if $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

Sol:- Since 'u' is not a homogenous f^n.

$u(x,y) = \tan u = \dfrac{x^3+y^3}{x-y}$.

by Euler's thm $\quad x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = nu.$

$$\Rightarrow x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2u.$$

$$\Rightarrow x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow x\cdot\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{1+\tan^2 u} = \sin 2u = g(u)$$

By Euler's thm of 2nd order, we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\left[g'(u)-1\right]$$

$$= \sin 2u\left[2\cos 2u - 1\right]$$

$$= 2\sin 2u\cos 2u - \sin 2u$$

$$= \sin 4u - \sin 2u$$

$$= 2\cos\left(\frac{4u+2u}{2}\right)\cdot\sin\left(\frac{4u-2u}{2}\right)$$

$$= 2\cos 3u\cdot\sin u.$$

$$\left[\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\cdot\sin\left(\frac{C-D}{2}\right)\right]$$

— * If $u = \sec^{-1}\left(\dfrac{x^3-y^3}{x+y}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2\cot u$, then

Evaluate $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$

(3). If (u,v) are functions of (r,s) and (r,s) are the functions of (x,y) then Jacobian of

$$J\left(\frac{u,v}{x,y}\right) = J\left(\frac{u,v}{r,s}\right) \cdot J\left(\frac{r,s}{x,y}\right)$$

Functional dependence :- let $u=u(x,y)$ & $v=v(x,y)$ are said to be functionally dependent, if $J\left[\frac{u,v}{x,y}\right]=0$

i.e.,
$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 0.$$

$\boxed{\text{Note}}$: (1). If $J\left[\frac{u,v}{x,y}\right] \neq 0$ then u & v are functionally independent.

(2). $u=u(x,y,z)$, $v=v(x,y,z)$ & $w=w(x,y,z)$ are functionally dependent if $J\left(\frac{u,v,w}{x,y,z}\right)=0$.

Problems :-

(1). $u=2x-3y$; $v=5x+4y$ then find $J\left(\frac{u,v}{x,y}\right)$.

Sol:-
$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2 & -3 \\ 5 & 4 \end{vmatrix} = 8+15 = 23 \neq 0.$$

$\therefore$ u,v are functionally independent.

(2). $u=x+y+z$, $v=y+z$, $w=z$ then find $J\left(\frac{u,v,w}{x,y,z}\right)$

$$\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0.$$

$\therefore$ functionally independent.

③. If $x = u$; $y = u \tan v$, $z = w$ then find $J\left(\frac{x, y, z}{u, v, w}\right)$ (23)

Sol:- $\begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ \tan v & u\sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u\sec^2 v \neq 0.$

$\therefore$ functionally independent.

**

④. $x = r\cos\theta$; $y = r\sin\theta$ then P.T $J\left(\frac{x, y}{r, \theta}\right) \cdot J\left(\frac{r, \theta}{x, y}\right) = 1$

Sol:- $J\left(\frac{x, y}{r, \theta}\right) = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$

$= r$ —— (1).

$J\left(\frac{r, \theta}{x, y}\right) = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix} = \begin{vmatrix} \frac{2x}{2\sqrt{x^2+y^2}} & \frac{2y}{2\sqrt{x^2+y^2}} \\ \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) & \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) \end{vmatrix}$ (W.K.T $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1} y/x$)

$= \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}}$

$= \frac{x^2+y^2}{(x^2+y^2)^{3/2}}$

$= \cancel{x^2+y^2} / \cancel{(x^2+y^2)}(x^2+y^2)^{1/2}$

$= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$ —— (2)

$\therefore$ (1) × (2)

$\Rightarrow J\left(\frac{x, y}{r, \theta}\right) \times J\left(\frac{r, \theta}{x, y}\right) = \cancel{r} \times \frac{1}{\cancel{r}}$

$= 1$

//

5. $u = x + 2y^3 - z^3$; $v = 2x^2yz$, $w = 2z^2 - xy$ then find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ at $(1,-1,0)$

Sol:- $\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & 6y^2 & -3z^2 \\ 4xyz & 2x^2z & 2x^2y \\ -y & -x & 4z \end{vmatrix}$

$= \begin{vmatrix} 1 & 6 & 0 \\ 0 & 0 & -2 \\ 1 & -1 & 0 \end{vmatrix}$

$= 1(-2) + 1(-12)$

$= -14$ //

— * If $x = u\sqrt{1-v^2} + v\sqrt{1-u^2}$ and $y = \sin^{-1}u + \sin^{-1}v$ then S.T 'x' and 'y' are functionally related. Also find the relationship.

6. $u = x(1-y)$ & $v = xy$ then find $J\left(\dfrac{u,v}{x,y}\right)$

Sol:- $J\left(\dfrac{u,v}{x,y}\right) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix} = x - \cancel{xy} + \cancel{xy} = x$ //

7. $x = \dfrac{u^2}{v}$; $y = \dfrac{v^2}{u}$ then find $J\left[\dfrac{u,v}{x,y}\right]$

Sol:- $J\left[\dfrac{u,v}{x,y}\right] = \dfrac{1}{J\left(\dfrac{x,y}{u,v}\right)}$ —— (1)

$\therefore J\left(\dfrac{x,y}{u,v}\right) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \dfrac{2u}{v} & \dfrac{-u^2}{v^2} \\ \dfrac{-v^2}{u^2} & \dfrac{2v}{u} \end{vmatrix} = \dfrac{4\cancel{uv}}{\cancel{vu}} - \dfrac{u^2\cancel{v^2}}{u^2\cancel{v^2}}$

$= 4 - 1$

$= 3.$

$(1) \Rightarrow J\left[\dfrac{u,v}{x,y}\right] = \dfrac{1}{3}$ //

(8). If $u=\frac{x+y}{1+xy}$, $v=\tan^{-1}x+\tan^{-1}y$ then P.T functionally dependent and find the relation between u & v.

Sol:- $J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$$u_x = \frac{(1-xy)-(x+y)(-y)}{(1-xy)^2} = \frac{1-\cancel{xy}+\cancel{xy}+y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}.$$

$$u_y = \frac{(1-xy)(1)-(x+y)(0-x)}{(1-xy)^2} = \frac{1+x^2}{(1+xy)^2}.$$

$$v_x = \frac{1}{1+x^2} \quad ; \quad v_y = \frac{1}{1+y^2}$$

$$\therefore J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0.$$

∴ u & v are functionally dependents.

W.K.T given that $v = \tan^{-1}x + \tan^{-1}y$

$$= \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$v = \tan^{-1}u$$

(or)

$$u = \tan v.$$

(9). If $u=\frac{x}{y}$ & $v=\frac{x+y}{x-y}$ then P.T u, v are functionally dependent & find the relation between them.

Sol:-

Sol: $J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$ —— (1)

$u_x = 1/y$; $u_y = -x/y^2$

$v_x = \dfrac{(x-y)(1)-(x+y)(1)}{(x-y)^2} = \dfrac{-2y}{(x-y)^2}$

$v_y = \dfrac{(x-y)(1)-(x+y)(-1)}{(x-y)^2} = \dfrac{2x}{(x-y)^2}$

$(1) \Rightarrow J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} 1/y & -x/y^2 \\ \frac{-2y}{(x-y)^2} & \frac{2x}{(x-y)^2} \end{vmatrix} = 0.$

$\therefore$ u, v are functionally dependent.

W.K.T. Relation between 'u & v' is $\boxed{\frac{u+1}{u-1} = v}$

(10) $u = x.e^y \sin z$; $v = xe^y \cos z$; $w = x^2 e^{2y}$ then P.T u, v, w are functionally dependent & find the relation.

Sol:- $J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} e^y \sin z & xe^y \sin z & xe^y \cos z \\ e^y \cos z & xe^y \cos z & -xe^y \sin z \\ 2xe^{2y} & 2x^2 e^{2y} & 0 \end{vmatrix}$

$= e^y e^y e^{2y} . 2x.x.x \begin{vmatrix} \sin z & \sin z & \cos z \\ \cos z & \cos z & -\sin z \\ 1 & 1 & 0 \end{vmatrix}$

$= \sin z(\sin z) - \sin z(\sin z) + \cos z.(\cos z - \cos z)$

$\sin^2 z - \sin^2 z + 0 = 0.$

$\therefore$ u, v, w are functionally dependent.

$\therefore$ Relation between u, v & w is $\boxed{u^2+v^2=w.}$

(11). $u = x+y+z$, $v = xy+yz+zx$, $\omega = x^2+y^2+z^2$ then P.T u, v, ω are functionally dependent and Find relation.

Sol:- $J\left(\frac{u,v,\omega}{x,y,z}\right) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ \omega_x & \omega_y & \omega_z \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 & 1 \\ y+z & x+z & y+x \\ 2x & 2y & 2z \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ y+z & x+z & y+x \\ x & y & z \end{vmatrix} = 0.$

$\therefore u, v, \omega$ are functionally dependent.

$\therefore$ Relation between u, v, ω is $\boxed{u^2 = \omega + 2v.}$

(12). $u = x+y+z$, $v = x^3+y^3+z^3-3xyz$, $\omega = x^2+y^2+z^2-xy-yz-zx$ then P.T u, v, ω are functionally dependent and find the relation.

Sol:- $J\left(\frac{u,v,\omega}{x,y,z}\right) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ \omega_x & \omega_y & \omega_z \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \\ 2x-y-z & 2y-x-z & 2z-y-x \end{vmatrix}$

$= 3\begin{vmatrix} 1 & 1 & 1 \\ x^2-yz & y^2-zx & z^2-xy \\ 2x-y-z & 2y-x-z & 2z-y-x \end{vmatrix}$

$C_3 = C_3 - C_2$; $C_2 = C_2 - C_1$

$= 3\begin{vmatrix} 1 & 0 & 0 \\ x^2-yz & y^2-zx-\cancel{x^2}+\cancel{yz} & z^2-xy-y^2+zx \\ 2x-y-z & 2y-x-z-2x+y+z & 2z-y-x-2y+x+z \end{vmatrix} = 3\begin{vmatrix} 1 & 0 & 0 \\ x^2-yz & y^2-zx-x^2+yz & z^2-xy-y^2+xz \\ 2x-y-z & 3y-3x & 3z-3y \end{vmatrix}$

$= 0.$

$\therefore u, v, \omega$ are functionally dependent.

$\therefore$ Relation between u, v, ω is $\boxed{v = \omega . u}$

Maxima & minima of a function with single variable:

— * Let $f(x)$ be a function if $\boxed{f'(x)=0}$ and $\boxed{f''(x)<0}$ then the function $f(x)$ is maximum at that point.

— * If $\boxed{f'(x)=0}$ and $\boxed{f''(x)>0}$ then the function $f(x)$ is minimum at that point.

— * If $\boxed{f'(x)=0}$ and $\boxed{f''(x)=0}$ then $f(x)$ is neither maximum nor minimum.

Maxima and minima of a function with two variables :—

necessary condition:- Let $f(x,y)$ be a function of two variables x & y

i.e, $\frac{\partial f}{\partial x}=0$ & $\frac{\partial f}{\partial y}=0$

— * Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ are equal to zero and solving the Eq's, we will get the stationary points as (a_1,b_1) (a_2,b_2) --- (a_n,b_n).

— * Find $r=\frac{\partial^2 f}{\partial x^2}$, $s=\frac{\partial^2 f}{\partial x \partial y}$, $t=\frac{\partial^2 f}{\partial y^2}$.

Sufficient Condition :—

— * Find r, s, t values at stationary points $(a_1,b_1), (a_2,b_2)$ --- (a_n,b_n). If

(i). $rt-s^2>0$ and $r<0$ then the function $f(x,y)$ is maximum and the maximum value at (a,b)

(ii). $rt-s^2>0$ and $r>0$ then the f^n $f(x,y)$ is minimum and the minimum value at (a,b)

(iii). $rt-s^2 < 0$ then the f^n $f(x,y)$ has neither maximum nor minimum at the point (a,b). Hence the point (a,b) called "Saddle point".

(iv). $rt-s^2=0$, No conclusion can be drawn about (x,y).

& find the respective maximum and minimum value of $f(x,y)$ at (a,b).

① Find maxima and minima of $f(x,y)=x^3+3xy^2-3x^2-3y^2+4$

Sol:- Given $f(x,y)=x^3+3xy^2-3x^2-3y^2+4$ —— (1)

Diff. eq^n (1) partially w.r.to 'x':

$$\frac{\partial f}{\partial x}=3x^2+3y^2-6x=0$$

$$\Rightarrow x^2+y^2-2x=0 \quad \text{—— (2)}$$

Diff. eq^n (1) partially w.r.to 'y'

$$\frac{\partial f}{\partial y}=6xy-6y=6(xy-y)=0$$

$$\Rightarrow y(x-1)=0$$

$$\Rightarrow \boxed{y=0\ ;\ x=1}$$

From eq^n (2), when $x=1$

$$1+y^2-2=0 \Rightarrow \boxed{y=\pm 1}$$

$$\therefore (x,y)=(1,1)\ \&\ (1,-1)$$

From eq^n (2) when $y=0$

$$\therefore x^2-2x=0 \Rightarrow x(x-2)=0 \Rightarrow x=0\ \&\ x=2$$

$$\therefore (x,y)=(0,0)\ \&\ (2,0)$$

$\therefore$ the stationary points are $(1,1)$, $(1,-1)$, $(0,0)$ & $(2,0)$

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(3x^2 + 3y^2 - 6x\right)$$

$$= 6x - 6 = 6(x-1)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[6xy - 6y\right] = \frac{\partial}{\partial x}\left[6(xy - y)\right] = 6y.$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[6xy - 6y\right] = 6x - 6.$$

→ at the point $(1,1) \Rightarrow r = 0;\ s = 6;\ t = 0$

$\therefore rt - s^2 = 0 - 36 < 0.$

the function $f(x,y)$ is no maximum and no minimum.

$\therefore (1,1)$ is a saddle point.

→ at the point $(1,-1) \Rightarrow r = 0;\ s = -6;\ t = 0$

$\therefore rt - s^2 = -36 < 0$

the f^n $f(x,y)$ is no maximum and no minimum at $(1,-1)$ $\therefore (1,-1)$ is a saddle point.

→ at the point $(0,0) \Rightarrow r = -6;\ s = 0;\ t = -6$

$\therefore rt - s^2 = 36 > 0$ & $r = -6 < 0.$

the f^n $f(x,y)$ is maximum at $(0,0)$.

$\therefore$ Find the maximum value of $f(x,y)$ at $(0,0)$ is

$\underline{f(0,0) = 4.}$

→ at the point $(2,0) \Rightarrow r = 6;\ s = 0;\ t = 6$

$\therefore rt - s^2 = 36 - 0^2 = 36 > 0$ & $r = 6 > 0$

$\therefore$ the f^n $f(x,y)$ is minimum at $(2,0)$ is

$f(2,0) = 0$ //. $\therefore$ $(1,1)$ & $(1,-1)$ = saddle points.

$(0,0), (2,0)$ = extreme points.

9. Find the maximum and minimum of $f(x,y) = xy(x+y-12)$. (28)

Sol:- $f(x,y) = xy(x+y-12)$

$f(x,y) = x^2y + xy^2 - 12xy$ —— (1)

$\frac{\partial f}{\partial x} = 2xy + y^2 - 12y = 0$ —— (2)

$\Rightarrow y(2x+y-12) = 0$.

$\frac{\partial f}{\partial y} = x^2 + 2xy - 12x = 0$. —— (3).

$\Rightarrow x(x+2y-12) = 0$

$x = 0$; $x+2y-12 = 0$

$x = 12-2y$.

From (2) when $\boxed{x=0}$.

$y(0+y-12) = 0$

$\boxed{y=0 ; y=12}$

From (3) when $y=0$.

$x(x+0-12) = 0$

$\boxed{x=0 ; x=12}$

$$\begin{array}{r} \cancel{2x} + y - 12 = 0 \\ -\cancel{2x} + 4y - 24 = 0 \\ \hline -3y + 12 = 0 \end{array}$$

$\boxed{y=4}$

$2x + 4 - 12 = 0$

$2x = 8$

$\boxed{x=4}$

$\therefore$ the stationary points are.

$(0,0)$, $(4,4)$, $(0,12)$, $(12,0)$.

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left[y(2x+y-12)\right]$$
$$= 2y.$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left[x(x+2y-12)\right]$$
$$= 2x+2y-12.$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$
$$= 2x.$$

→ At the point $(0,0)$:- $rt - s^2$
$= -144 < 0.$

∴ the function $f(x,y)$ is neither maximum nor minimum and the point $(0,0)$ is called "saddle point."

→ At the point $(0,12)$:- $rt - s^2$
$= -144 < 0.$

∴ the function $f(x,y)$ is neither maximum nor minimum and the point $(0,12)$ is called "saddle point."

→ At the point $(12,0)$:- $rt - s^2$
$= -144 < 0.$

∴ the function $f(x,y)$ is neither maximum nor minimum and the point $(12,0)$ is called "saddle point".

→ At the point $(4,4)$:- $rt - s^2$
$= 48 > 0.$ & $r = 2(4)$
$= 8 > 0.$

∴ The function $f(x,y)$ is minimum at $(4,4)$ & the minimum value is

$f(x,y) = f(4,4) = xy(x+y-12) = -32$ //.

Def:- Let $f(x,y)$ be a function of two variables 'x' & 'y' (30).
at $x=a$, $y=b$ then $f(x,y)$ is said to have maximum (or) minimum value (extreme value).

If $f(a,b) > f(a+h, b+k)$ (or)

$f(a,b) < f(a+h, b+k)$.

where h & k are small values.

Extreme value:- $f(a,b)$ is said to be an extreme value of 'f'. If it is a maximum (or) minimum value.

stationary value:-

$f(a,b)$ is said to be a stationary value of $f(x,y)$

if $f_x(a,b)=0$

$f_y(a,b)=0$

thus Every extreme value is a stationary value, But the converse may not be true.

(3). Find the extreme values of

$f(x,y) = x^2-y^2+6x-12$. — (1)

Sol. $\frac{\partial f}{\partial x}=0 \Rightarrow 2x+6=0 \Rightarrow \boxed{x=-3}$ (2)

$\frac{\partial f}{\partial y}=0 \Rightarrow -2y=0 \Rightarrow \boxed{y=0}$ (3)

$\therefore$ The stationary point is $(-3,0)$

$r = \frac{\partial^2 f}{\partial x^2} = 2.$

$s = \frac{\partial^2 f}{\partial x \partial y} = 0.$

$t = \frac{\partial^2 f}{\partial y^2} = -2.$

$\therefore rt - s^2 = -4 < 0.$

$\therefore$ the function is having neither maximum nor minimum

$\therefore (-3,0)$ is a saddle point.

**** ④. Find the extreme value of $f(x,y) = x^3 + y^3 - 3axy$ $(a>0)$.**

Sol:- $\frac{\partial f}{\partial x} = 3x^2 - 3ay = 0$ —— (1) $\Rightarrow \frac{y^4}{a^2} - ay = 0$

$3(x^2 - ay) = 0$

$\frac{\partial f}{\partial y} = 3y^2 - 3ax = 0$ —— (2). $\Rightarrow y^4 - a^3 y = 0$

$y^2 - ax = 0.$ $\Rightarrow y = 0;\ y^3 - a^3 = 0$

$y^2 = ax.$ $y = a.$

$x = \frac{y^2}{a}$

$y = 0 \Rightarrow x = 0.$

$y = a \Rightarrow x = a$

$\therefore (0,0)$ & (a,a). are stationary points.

$r = \frac{\partial^2 f}{\partial x^2} = 6x$

$s = \frac{\partial^2 f}{\partial x \partial y} = -3a$

$t = \frac{\partial^2 f}{\partial y^2} = 6y$

At $(0,0)$:- $rt - s^2 = -9a^2 < 0.$

$\therefore f(x,y)$ is neither maximum nor minimum

$\therefore (0,0)$ is saddle point.

At (a,a) :- $rt - s^2 = 27a^2 > 0$, $r = 6a > 0$

$\therefore f(a,a)$ is minimum at (a,a).

(a, a) is extreme point.

minimum value at $(a, a) = -a^3$ //.

5. Find the extreme values of $\sin x + \sin y + \sin(x+y)$ — (1)

Sol:- $\frac{\partial f}{\partial x} = 0 \Rightarrow \cos x + \cos(x+y) = 0.$ — (2).

$\frac{\partial f}{\partial y} = 0 \Rightarrow \cos y + \cos(x+y) = 0$ — (3)

solving (2) & (3)

$\boxed{x = y}$.

(2) $\Rightarrow 2\cos\left(\frac{2x+y}{2}\right) \cdot \cos\left(\frac{-y}{2}\right) = 0.$

$\Rightarrow 2\cos\left(\frac{2x+y}{2}\right) \cdot \cos(y/2) = 0.$

$\Rightarrow 2\cos\left(\frac{3x}{2}\right)\cos(x/2) = 0 \quad (\because x = y)$

$\cos\frac{3x}{2} = 0$ $\qquad$ $\cos x/2 = 0.$

$\frac{3x}{2} = \pi/2$ $\qquad$ $x/2 = \pi/2$

$\boxed{x = \pi/3}$ $\qquad$ $\boxed{x = \pi}$.

If $x = \pm\pi/3 \Rightarrow y = \pm\pi/3.$

$x = \pi \Rightarrow y = \pi.$

$\therefore$ the stationary points are $(\pi/3, \pi/3)$, $(-\pi/3, -\pi/3)$
(π, π), $(-\pi, \pi)$.

$r = \frac{\partial^2 f}{\partial x^2} = -[\sin x + \sin(x+y)]$

$s = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$

$t = \frac{\partial^2 f}{\partial y^2} = -[\sin y + \sin(x+y)]$

At point $(\pi/3, \pi/3) = rt - s^2$

$= 9/4 > 0.$

$\& \; r = -\sqrt{3} < 0$

$\therefore f(x,y)$ is maximum at $(\pi/3, \pi/3)$.

$\therefore f(x,y) = \frac{3\sqrt{3}}{2}$ = maximum value $\quad \therefore (\pi/3, \pi/3)$ is a extreme value.

At the point $(-\pi/3, -\pi/3)$:- $rt - s^2$

$= 9/4 > 0.$

$\& \; r = \sqrt{3} > 0.$

$\therefore f(x,y)$ is minimum at $(-\pi/3, -\pi/3)$

$\therefore f(x,y) = \frac{-3\sqrt{3}}{2}$ = minimum value. $\quad \therefore (-\pi/3, -\pi/3)$ is a extreme value.

At the Point $(\pi, \pi) = rt - s^2$

$= 0.$

At the point $(-\pi, \pi) = rt - s^2$

$= 0$

Lagrange's method of undetermined multiplier :-

Let it is required to find the extremum of $f(x,y,z)$ subject to the condition $\phi(x,y,z) = 0$ —(1).

(1). Take Lagrange's function as.

$$F(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z).$$

where 'λ' is lagrange's undefined multiplier.

(2): Find the Eq^{n}s

$$\frac{\partial F}{\partial x} = F_x = f_x + \lambda \phi_x = 0 \quad \text{---(2)}$$

$$\frac{\partial F}{\partial y} = F_y = f_y + \lambda \phi_y = 0 \quad \text{---(3)}$$

$$\frac{\partial F}{\partial z} = F_z = f_z + \lambda \phi_z = 0 \quad \text{---(4)}$$

(3). Solve Eq^{n}s (1), (2), (3) & (4) and find x, y, z values which gives extremum values.

(4). Find the respective maximum (or) minimum at that points.

(1). Find the Extremum values of $x+y+z$ subject to $1/x + 1/y + 1/z = 1$.

Sol^n:- Let $P(x,y,z)$ be a point on the plane then

$$\phi(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \quad \text{---(1)}.$$

Let $f(x,y,z) = x+y+z$.

$\therefore$ the Lagrange's function

$$F(x,y,z) = f(x,y,z) + \lambda \, \phi(x,y,z)$$
$$= (x+y+z) + \lambda\left(1/x + 1/y + 1/z - 1\right)$$

Differentiate 'F' partially w.r.to x, y, z

$$\Rightarrow \frac{\partial F}{\partial x} = F_x = (1) + \lambda\left(\frac{-1}{x^2}\right) = 0 \rightarrow 1 - \frac{\lambda}{x^2} = 0 \Rightarrow \lambda = x^2$$

$$\Rightarrow \boxed{x = \pm\sqrt{\lambda}}.$$

$\Rightarrow F_y = \frac{\partial F}{\partial y} = 1 + \lambda(-1/y^2) = 0 \Rightarrow \boxed{y = \pm\sqrt{\lambda}}$

$\Rightarrow F_z = \frac{\partial F}{\partial z} = 1 + \lambda(-1/z^2) = 0 \Rightarrow \boxed{z = \pm\sqrt{\lambda}}$

sub" x, y & z values in (1)

$\phi(x,y,z) = 1/\sqrt{\lambda} + 1/\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}} = 1$

$\Rightarrow \frac{3}{\sqrt{\lambda}} = 1$

$\Rightarrow \boxed{\lambda = 9}$

∴ The stationary points are $(3,3,3)$, $(-3,-3,-3)$.

∴ $(3,3,3)$ & $(-3,-3,-3)$ ~~are~~ only satisfies Eq^n(1).

∴ $F(x,y,z) = (3+3+3) + \lambda(1/3 + 1/3 + 1/3 - 1)$ & $\underset{(min)}{F(x,y,z) = -9}$ (put $x=-3$, $y=-3$, $z=-3$ in (1))

$= \underset{(max)}{9}$ // ∴ The maximum and minimum values are 9 & -9.

② Find the maximum value $u = x^2y^3z^4$ subject to $2x + 3y + 4z = 18$.

Sol:- Let $P(x,y,z)$ be a point on the line then

$\phi(x,y,z) = 2x + 3y + 4z = 18$ —— (1)

Given $u(x,y,z) = x^2y^3z^4$.

∴ the Lagrange's function

$F(x,y,z) = u(x,y,z) + \lambda\phi(x,y,z)$

$\Rightarrow F(x,y,z) = x^2y^3z^4 + \lambda(2x + 3y + 4z - 18)$

Diff" 'F' partially w.r.to x, y, & z.

$\Rightarrow F_x = \frac{\partial F}{\partial x} = 2xy^3z^4 + 2\lambda = 0$ & $F_y = \frac{\partial F}{\partial y} = 3x^2y^2z^4 + 3\lambda = 0$

$\Rightarrow xy^3z^4 = -\lambda$ $\Rightarrow 3x^2y^2z^4 = -3\lambda$

$\Rightarrow x^2y^3z^4 = -\lambda x$ $\Rightarrow x^2y^2z^4 \cdot y = \cancel{3}\lambda y$

$\Rightarrow u = -\lambda x$ $\Rightarrow u = -\lambda y$

$\Rightarrow \boxed{x = -\frac{u}{\lambda}}$ $\Rightarrow \boxed{y = -\frac{u}{\lambda}}$

$\Rightarrow F_z = \frac{\partial f}{\partial z} = 4x^2y^3z^3 + 4\lambda = 0$

$\Rightarrow x^2y^3z^3 = -\lambda$

$\Rightarrow u\cancel{z} = -\lambda z$

$\Rightarrow \boxed{z = -\frac{u}{\lambda}}$

Subn x, y & z values in (1).

$\therefore \phi(x,y,z) = 2x+3y+4z-18 = 0$

$\Rightarrow 2\left(\frac{-u}{\lambda}\right) + 3\left(\frac{-u}{\lambda}\right) + 4\left(\frac{-u}{\lambda}\right) - 18 = 0$

$\Rightarrow -\frac{2u}{\lambda} - \frac{3u}{\lambda} - \frac{4u}{\lambda} = 18$

$\Rightarrow \frac{-\cancel{9}u}{\lambda} = \cancel{18}^{2}$

$\Rightarrow \frac{u}{\lambda} = -2 \Rightarrow \boxed{\lambda = -\frac{u}{2}}$

$\therefore x = \frac{-u}{\left(\frac{-u}{2}\right)} = 2$; $y = \frac{-u}{\left(-u/2\right)} = 2$; $\boxed{z = 2}$

$\boxed{y = z = x}$.

The stationary point is $(2, 2, 2)$

$\therefore f(x,y,z)$ is maximum at $(2,2,2)$ $\therefore f(x,y,z)_{(2,2,2)} = 512$ //

(or), $F(2,2,2) = 2^2 2^3 2^4 + \lambda[4+\cancel{2}+6+8-18] = 512$ //

3. Find the point on the plane $2x+3y-z=5$, which is nearest to the origin.

Sol:- Let $P(x,y,z)$ be a point on the plane.

$$\phi(x,y,z) = 2x+3y-z-5 = 0 \quad \text{--- (1)}$$

the distance between the origin and the point $P(x,y,z)$

is $OP = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2} = \sqrt{x^2+y^2+z^2}$.

∴ we have to minimize the function

$f(x,y,z) = x^2+y^2+z^2$, subject to the condition $2x+3y-z-5=0$.

is ∴ the Lagrange's function is

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$$

$$F(x,y,z) = (x^2+y^2+z^2) + \lambda(2x+3y-z-5) \quad \text{--- (2)}$$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x+2\lambda = 0$; $\frac{\partial F}{\partial y} = 0 \Rightarrow 2y+3\lambda = 0$; $\frac{\partial F}{\partial z} = 0$

$\Rightarrow \boxed{x=-\lambda}$; $\Rightarrow \boxed{y = \frac{-3\lambda}{2}}$; $\Rightarrow 2z-\lambda = 0$

$\Rightarrow \boxed{z = \lambda/2}$

Subu $x=-\lambda$; $y = -\frac{3}{2}\lambda$; $z = \lambda/2$ in (1).

$$\phi(x,y,z) = 2x+3y-z-5 = 0$$

$$\boxed{\Rightarrow \lambda = -5/7}$$

∴ $\boxed{x = 5/7}$; $\boxed{z = \frac{-5}{14}}$; $\boxed{y = \frac{15}{14}}$

∴ x, y, z values in $f(x,y,z) = x^2+y^2+z^2$

$= 25/14$.

∴ The minimum value is $25/14$ at the point $\left(5/7, \frac{15}{14}, \frac{-5}{14}\right)$.

④. Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipse solid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sol: Let the measurements of the parallelopiped be $2x, 2y, 2z$.

So, i.e. the volume of the rectangular parallelopiped $= 8xyz$

Now we have to maximize $f(x,y,z) = 8xyz$

subject to the condition $\phi(x,y,z) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ——(1)

∴ Lagrange's function

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$$

$$= 8xyz + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$$

$\frac{\partial F}{\partial x} = 8yz + \lambda\left(\frac{2x}{a^2}\right) = 0$

$\Rightarrow \lambda\left(\frac{2x}{a^2}\right) = -8yz$

$\Rightarrow x = \frac{-4yza^2}{\lambda}$

$\Rightarrow \frac{\lambda}{-4} = \frac{yza^2}{x}$ ——(2)

; $\frac{\partial F}{\partial y} = 8xz + \frac{\lambda}{b^2}(2y) = 0$

$\Rightarrow \frac{\lambda}{-4} = \frac{xzb^2}{y}$ ——(3)

$\Vert^y$ $\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\lambda}{-4} = \frac{xyc^2}{z}$ ——(4)

(2) = (3)

$\frac{x\cancel{z}b^2}{y} = \frac{y\cancel{z}a^2}{x}$

$\Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2}$ ——(5)

(3) = (4)

$\frac{\cancel{x}zb^2}{y} = \frac{\cancel{x}yc^2}{z}$

$\Rightarrow \frac{b^2}{y^2} = \frac{c^2}{z^2}$ ——(6)

$\Rightarrow \frac{z^2}{c^2} = \frac{y^2}{b^2}$

from (5) & (6)

$\therefore \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$. Here $x^2/a^2 = y^2/b^2 = z^2/c^2$.

Subn in eqn (1).

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \Rightarrow \frac{x^2}{a^2}+\frac{x^2}{a^2}+\frac{x^2}{a^2}=1$$

$$\Rightarrow \boxed{x=\pm a/\sqrt{3}}$$

$\|^y$ $\frac{y^2}{b^2}+\frac{y^2}{b^2}+\frac{y^2}{b^2}=1 \Rightarrow \boxed{y=\pm b/\sqrt{3}}$

$\|^y$ $\frac{z^2}{a^2}+\frac{z^2}{b^2}+\frac{z^2}{c^2}=1 \Rightarrow \boxed{z=\pm c/\sqrt{3}}$

$\therefore$ The stationary points are $(a/\sqrt{3}, b/\sqrt{3}, c/\sqrt{3})$ & $(a/\sqrt{3}, -b/\sqrt{3}, -c/\sqrt{3})$

If at the point $(a/\sqrt{3}, b/\sqrt{3}, c/\sqrt{3})$ then

$f(x,y,z) = \frac{8abc}{3\sqrt{3}}$ $\therefore$ It is the maximum value of $f(x,y,z)$.

If at the point $(-a/\sqrt{3}, -b/\sqrt{3}, -c/\sqrt{3})$ then

$f(x,y,z) = \frac{-8abc}{3\sqrt{3}}$ $\therefore$ It is the minimum value of $f(x,y,z)$

(5). Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsed solid $4x^2+4y^2+4z^2=36$.

Sol:- Let the measurements of the rectangular parallelopiped be $2x, 2y, 2z$. so, i.e, the volume of the rectangular parallelopiped $= 8xyz$.

Now we have to maxize $f(x,y,z) = 8xyz$

subject to the condition $\phi(x,y,z) = 4x^2+4y^2+4z^2=36$ — (1)

$\therefore$ Lagrange's function (40)

$$F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$$

$$F(x,y,z) = 8xyz + \lambda\left(4x^2+4y^2+4z^2-36\right) \quad \text{--- (2)}$$

$$\frac{\partial F}{\partial x} = 8yz + 8\lambda x = 0 \Rightarrow \frac{\partial F}{\partial x} = 8(yz+\lambda x) = 0$$

$$\Rightarrow yz + \lambda x = 0$$

$$\Rightarrow yz = -\lambda x$$

$$\Rightarrow \lambda = -\frac{yz}{x} \quad \text{--- (3)}$$

$$\frac{\partial F}{\partial y} = 0 \text{ lly } \lambda = -\frac{zx}{y} \quad \text{--- (4)}$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow \lambda = -\frac{xy}{z} \quad \text{--- (5)}$$

(3) = (4).

$$-\frac{y\cancel{z}}{x} = -\frac{\cancel{z}x}{y} \Rightarrow \boxed{x^2=y^2}$$

&

(4) = (5)

$$-\frac{z\cancel{x}}{y} = -\frac{\cancel{x}y}{z} \Rightarrow \boxed{z^2=y^2}$$

(5) = (3)

$$-\frac{x\cancel{y}}{z} = -\frac{\cancel{y}z}{x} \Rightarrow \boxed{x^2=z^2}$$

$$\therefore \boxed{x^2=y^2=z^2}$$

subs in eqn (1)

$$4x^2+4x^2+4x^2 = 36 \Rightarrow 12x^2 = 36 \Rightarrow \boxed{x = \pm\sqrt{3}}$$

$$\boxed{y = \pm\sqrt{3}}$$

$$\boxed{z = \pm\sqrt{3}}$$

$\therefore$ the stationary points are $(\sqrt{3},\sqrt{3},\sqrt{3})$ & $(-\sqrt{3},-\sqrt{3},-\sqrt{3})$.

at $(\sqrt{3},\sqrt{3},\sqrt{3})$:- $f(x,y,z) = 8xyz = 8\sqrt{3}.\sqrt{3}.\sqrt{3} = 24\sqrt{3}$ = maximum

at $(-\sqrt{3},-\sqrt{3},-\sqrt{3})$:- $f(x,y,z) = 8(-\sqrt{3})(-\sqrt{3})(-\sqrt{3}) = -24\sqrt{3}$ = minimum

(41)

⑥. $u = x^4+y^4+z^4$ the condition is subject to $xyz = a^3$.

Sol:- $\phi(x,y,z) = xyz - a^3 = 0$ — (1)

$f(x,y,z) = x^4+y^4+z^4$

$\therefore$ The Lagrange's function is

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

$= (x^4+y^4+z^4) + \lambda(xyz - a^3)$ — (2)

$\frac{\partial F}{\partial x} = 4x^3 + \lambda yz = 0 \Rightarrow 4x^3 = -\lambda yz$.

$\Rightarrow -\frac{\lambda}{4} = \frac{x^3}{yz}$ — (3).

$\frac{\partial F}{\partial y} = 4y^3 + \lambda xz = 0 \Rightarrow 4y^3 = -\lambda xz$

$\Rightarrow -\frac{\lambda}{4} = \frac{y^3}{xz}$ — (4).

$\frac{\partial F}{\partial z} = 4z^3 + \lambda xy = 0 \Rightarrow 4z^3 = -\lambda xy$

$\Rightarrow -\frac{\lambda}{4} = \frac{z^3}{xy}$ — (5).

(3) = (4).

$\frac{x^3}{y\cancel{z}} = \frac{y^3}{x\cancel{z}} \Rightarrow \boxed{x^4 = y^4}$ $\boxed{x = y}$

; (4) = (5)

$\frac{y^3}{\cancel{x}z} = \frac{z^3}{\cancel{x}y} \Rightarrow \boxed{y^4 = z^4}$ $\boxed{y = z}$

; (5) = (3)

$\frac{z^3}{xy} = \frac{x^3}{yz}$

$\Rightarrow \boxed{z^4 = x^4}$ $\boxed{z = x}$

sub. in (1).

$\to x.x.x = a^3$

$\Rightarrow x^3 = a^3$

$\Rightarrow \boxed{x = a}$

Ily $\boxed{y = a}$ Ily $\boxed{z = a}$

$\therefore$ The stationary points are (a,a,a).

$\therefore$ at (a,a,a) :- $f(x,y,z) = a^4+a^4+a^4 = 3a^4$ = maximum

(7) Find the rectangular parallelopiped of maximum volume that can be inscribed in a sphere.

(or).

S.T the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.

Sol:- Let 'a' (constant) be the radius of the given sphere. Also let x, y, z be the length, breadth and height of a rectangular parallelopiped inscribed in the given sphere.

The Eqn of the sphere is $x^2+y^2+z^2=a^2$ — (1)

$\phi(x,y,z) =$

Volume of the rectangular parallelopiped is

$f = V = xyz$.

∴ Lagrange's function is

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$ — (2)

$= xyz + \lambda(x^2+y^2+z^2-a^2)$

$\frac{\partial f}{\partial x} = yz + 2\lambda x = 0 \Rightarrow yz = -2\lambda x \Rightarrow -2\lambda = \frac{yz}{x}$ — (3)

$\frac{\partial f}{\partial y} = xz + 2\lambda y = 0 \Rightarrow xz = -2\lambda y \Rightarrow -2\lambda = \frac{xz}{y}$ — (4)

$\frac{\partial f}{\partial z} = xy + 2z\lambda = 0 \Rightarrow xy = -2\lambda z \Rightarrow -2\lambda = \frac{xy}{z}$ — (5)

(3) = (4) & (4) = (5) & (5) = (3).

$\frac{y\cancel{z}}{x} = \frac{x\cancel{z}}{y}$ $\quad$ $\frac{\cancel{x}z}{y} = \frac{\cancel{x}y}{z}$ $\quad$ $\frac{xy}{z} = \frac{yz}{x}$

$\rightarrow \boxed{y^2=x^2}$ $\quad$ $\boxed{z^2=y^2}$ $\quad$ $\boxed{x^2=z^2}$

sub. in (1) $\rightarrow 3x^2 = a^2 \Rightarrow \boxed{x=\pm\frac{a}{\sqrt{3}}}$; $\boxed{y=\pm a/\sqrt{3}}$; $\boxed{z=\pm a/\sqrt{3}}$

∴ volume is maximum at $(a/\sqrt{3}, a/\sqrt{3}, a/\sqrt{3})$ is $V = xyz = \frac{a^3}{3\sqrt{3}}$ //.

8. Divide 24 into three parts such that the continued product of first, square of the second and cube of the third is max.

Sol:- Let '24' be divided into three parts x, y, z

then $x+y+z=24$ —— (1) $\phi(x,y,z)$

Take $f(x,y,z) = zy^2x^3$ —— (2).

∴ Lagrange's function is

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

$= x^3y^2z + \lambda(x+y+z-24)$

$\frac{\partial F}{\partial x} = 3x^2y^2z + \lambda = 0 \Rightarrow \lambda = -3x^2y^2z \Rightarrow \frac{\lambda x}{-3} = x^3y^2z \Rightarrow \frac{\lambda x}{-3} = f \Rightarrow \boxed{x = \frac{-3f}{\lambda}}$

$\frac{\partial F}{\partial y} = 3yx^3z + \lambda = 0 \Rightarrow \frac{\lambda}{-3} = yx^3z \Rightarrow \frac{\lambda y}{-3} = x^3y^2z \Rightarrow \frac{\lambda y}{-3} = f \Rightarrow \boxed{y = \frac{-3f}{\lambda}}$

$\frac{\partial F}{\partial z} = x^3y^2 + \lambda = 0 \Rightarrow \lambda = -x^3y^2 \Rightarrow \lambda z = -y^2x^3z \Rightarrow \boxed{z = \frac{-f}{\lambda}}$

(2) $\Rightarrow \phi(x,y,z) = \frac{-3f}{\lambda} - \frac{3f}{\lambda} - \frac{f}{\lambda} = 24$

$\Rightarrow \frac{-7f}{\lambda} = 24 \Rightarrow \lambda = -\frac{7f}{24} \Rightarrow \boxed{\lambda = \frac{-7f}{24}}$

∴ $x = \frac{-3f}{\lambda} = \frac{-3f}{\left(\frac{-7f}{24}\right)} = \frac{-3\cancel{f} \times 24}{-7\cancel{f}} = \frac{72}{7}$

$y = \frac{-3f}{\lambda} \Rightarrow y = \frac{72}{7}$

$z = \frac{-\cancel{f} \times 24}{-7\cancel{f}} = 24/7.$

∴ $f(x,y,z) = \left(\frac{72}{7}\right)^3\left(\frac{72}{7}\right)^2\left(\frac{24}{7}\right) = (10.28)^3(10.28)^2(3.42)$

$= (105.67)(3.42)$

$= 361.39$

9. Find the maximum and minimum distances of the point $(3,4,12)$ from the sphere $x^2+y^2+z^2=1$.

Sol:- $AP^2 = (x-3)^2 + (y-4)^2 + (z-12)^2 = f(x,y,z)$

$\phi(x,y,z) = x^2+y^2+z^2-1 = 0$. $\left(3/13, 4/13, 12/13\right)$ & $\left(-3/13, -4/13, -12/13\right)$

(max = 14 ; min = 12)

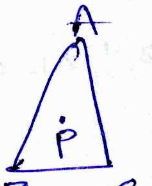
10. Find the maximum value of $u = x^2 y^3 z^4$ if $2x+3y+4z=a$.

$\left[\therefore \text{max} = \left(\frac{a}{9}\right)^9\right]$

11. Find a point within a triangle such that the sum of the squares of its distances from the three vertices is a minimum.

Sol:- Let $A(x_1, y_1)$, $B(x_2, y_2)$ & $C(x_3, y_3)$ be the vertices of ΔABC.

Also let $P(x,y)$ be a point in the ΔABC.



$\Rightarrow f(x,y) = AP^2 + BP^2 + CP^2$

$= (x-x_1)^2 + (y-y_1)^2 + (x-x_2)^2 + (y-y_2)^2 + (x-x_3)^2 + (y-y_3)^2$

$f(x,y) = \sum_{i=1}^{3}\left[(x-x_i)^2 + (y-y_i)^2\right]$

$\Rightarrow \frac{\partial f}{\partial x} = 0 \Rightarrow \sum_{i=1}^{3}(x-x_i) = 0. \Rightarrow (x-x_1)+(x-x_2)+(x-x_3) = 0$

$\Rightarrow 3x - (x_1+x_2+x_3) = 0 \Rightarrow x = \frac{x_1+x_2+x_3}{3}$

$\frac{\partial f}{\partial y} = 0 \Rightarrow \sum_{i=1}^{3}(y-y_i) = 0 \Rightarrow (y-y_1)+(y-y_2)+(y-y_3) = 0$

$\Rightarrow 3y - (y_1+y_2+y_3) = 0. \Rightarrow y = \frac{y_1+y_2+y_3}{3}$

Now $r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left[2(x-x_1)+2(x-x_2)+2(x-x_3)\right]$

$= \frac{\partial}{\partial x}\left\{2\left[3x-(x_1+x_2+x_3)\right]\right\}$

$= 6.$

$s = \frac{\partial^2 f}{\partial x \partial y} = 0$

$t = \frac{\partial^2 f}{\partial y^2}$ 6.

Now $rt - s^2 = 36$ & $r = 6 > 0$.
> 0

$\therefore f(x,y)$ is minimum for $x = \frac{x_1+x_2+x_3}{3}$; $y = \frac{y_1+y_2+y_3}{3}$.

Hence the required point is $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$,

which is the Centroid of ΔABC //

* Find the maximum and minimum values of the function $f(x,y) = x^3y^2(1-x-y)$

Sol: $f(x,y) = x^3y^2(1-x-y) = x^3y^2 - x^3y^2x - x^3y^3 = x^3y^2 - x^4y^2 - x^3y^3$

$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = x^2y^2(3-4x-3y) = 0$

$\Rightarrow x^2y^2(3-4x-3y) = 0$ — (1)

$\frac{\partial f}{\partial y} = 0 \Rightarrow 2x^3y - 2x^4y - 3x^3y^2 = 0$

$\Rightarrow x^3y(2-2x-3y) = 0$ — (2)

$\Rightarrow \boxed{x=0}$; $\boxed{y=0}$; $2-2x-3y=0$

$\Rightarrow \boxed{3y = 2-2x}$

at $\boxed{3y = 2-2x}$ in (1)

$\boxed{x=0}$ & $3-4x-2+2x=0$

$1-2x=0 \Rightarrow \boxed{x = 1/2}$

at $\boxed{x=0} \Rightarrow \boxed{y = 2/3}$

at $\boxed{x=1/2} \Rightarrow y = \frac{2-1}{3} \Rightarrow \boxed{y = 1/3}$

$\therefore (1/2, 1/3)$ & $(0, 2/3)$.

at $x=0 \Rightarrow x=0$; $\boxed{y=0}$; $3-3y=0$

$\Rightarrow \boxed{y=1}$

$\therefore$ points are $(0,0)$, $(0,1)$

at $y=0 \Rightarrow \boxed{x=0}$; $y=0$; $3-4x=0$

$\Rightarrow \boxed{x = 3/4}$ $\therefore$ points are $(0,0)$, $(3/4, 0)$ (A: $-1/432$)

— * Find the maximum and minimum values of

$f(x,y) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$

A:- 2 //.